Dynamic Free Riding with Irreversible Investments: On-line Appendix

Abstract
In this appendix we present the proofs omitted in “Dynamic Free Riding with Irreversible Investments” by Marco Battaglini, Salvatore Nunnari and Thomas Palfrey.

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1 Proof of Proposition 1

Proposition 1. For any $d$, $\delta$, $n$ and $y^o \in \left[ [u']^{-1} (1 - \delta(1 - d)), [u']^{-1} (1 - \delta(1 - d/n)) \right]$, there is an equilibrium with steady state $y^o$ in an irreversible economy. In all these equilibria convergence is monotonic and gradual.

Define $y^*(\delta, d, n) = [u']^{-1} (1 - \delta(1 - d)/n)$ and $y^{**}(\delta, d, n) = [u']^{-1} (1 - \delta(1 - d/n))$: these are the points at which

$$y'(g) = \frac{1 - d - \frac{n(1-u'(g))}{\delta}}{1 - n}$$

(1)

is, respectively, zero and one. Define $\gamma(d, \delta) = [u']^{-1} (1 - \delta(1 - d))$: this is the point at which (1) is equal to 1 − $d$. Note that $y^*(\delta, d, n) < \gamma(d, \delta)$ and $\gamma(d, \delta) < y^{**}(\delta, d, n)$. Moreover, since we are assuming that the planner interior solution is feasible ($y^*_p(\delta, d, n) < W/d$), we have $y^{**}(\delta, d, n) < W/d$. To construct an equilibrium with steady state $y^o \in [\gamma(\delta, d), y^{**}(\delta, d, n)]$ we proceed in 3 steps.

Step 1. We first construct the strategies associated to a generic $y^o$. For a generic $y^o \in [\gamma(\delta, d), y^{**}(\delta, d, n)]$, let $\tilde{y}(g | y^o)$ be the solution of the differential equation (1) when we require the initial condition: $\tilde{y}(g^o | y^o) = y^o$. Given $y^o$, moreover, let us define the two thresholds $g^3(y^o) = y^o/(1 - d)$ and $g^2(y^o) = \max \{ \min_{g > 0} \{ g | \tilde{y}(g | y^o) \leq W + (1 - d)g \} , y^*(\delta, d, n) \}$. In words, the second threshold is the largest point between the point at which $\tilde{y}(g | y^o)$ crosses from below $W + (1 - d)g$, and $y^*(\delta, d, n)$ (see Figure 1 in the paper for an example). It is easy to verify that, by construction, $g^3(y^o) \geq \gamma(\delta, d)$; moreover, $\tilde{y}(g | y^o) \in ((1-d)g, W + (1-d)g)$ with $\tilde{y}'(g | y^o) \in [0,1]$ and $\tilde{y}''(g | y^o) \geq 0$ in $[g^2(y^o), y^o)$. For any $y^o \in [\gamma(\delta, d), y^{**}(\delta, d, n)]$, we now define the investment function as follows:

$$y(g | y^o) = \begin{cases} 
\min \left\{ W + (1 - d)g, \tilde{y}(g^2(y^o) | y^o) \right\} & g \leq g^2(y^o) \\
\tilde{y}(g | y^o) & g^2(y^o) < g \leq y^o \\
y^o & y^o < g \leq g^3(y^o) \\
(1 - d)g & g > g^3(y^o) 
\end{cases}$$

(2)

Note that when depreciation is zero, then $g^2(y^o) = y^o$ and $y(g | y^o) = 1$ at $g = y^o$: so (2) coincides exactly with the investment function illustrated in Figure 1 in the paper. For future reference, define $g^1(y^o) = \max \{ 0, (\tilde{y}(g^2(y) | y^o) - W) / (1 - d) \}$. This is the point at which $W + (1 - d)g = \tilde{y}(g^2(y^o) | y^o)$, if positive. Since $\tilde{y}(g^2(y) | y^o) < W + (1 - d)g^2(y^o)$, $g^1(y^o) \in [0, g^2(y^o)]$. We have:

Lemma A.1. $y(g | y^o) \in [g^2(y^o), y^o]$ for $g \in [g^2(y^o), y^o]$.
Proof. Because \( y(g|y^o) \) is monotonic non-decreasing in \( g \in [g^2(y^o), y^o] \), for any \( g \in [g^2(y^o), y^o] \) we have \( y(g|y^o) \in [y(g^2(y^o)|y^o), y^o] \). Since \( y(g|y^o) \) has slope lower than one in \( [g^2(y^o), y^o] \) and \( y(y^o|y^o) = y^o \) for \( y^o \geq g^2(y^o) \), we must have \( y(g^2(y^o)|y^o) \geq g^2(y^o) \), so \( y(g|y^o) \geq g^2(y^o) \) for \( g \in [g^2(y^o), y^o] \). Similarly, \( y(y^o|y^o) = y^o \) implies \( y(g|y^o) \leq y^o \) for \( g \in [g^2(y^o), y^o] \). ■

Step 2. We now construct the value functions corresponding to each steady state \( y^o \). For \( g \in [g^2(y^o), y^o] \) define the value function recursively as

\[
v(g|y^o) = \frac{W + (1 - d)g - y(g|y^o)}{n} + u(y(g|y^o)) + \delta v(y(g|y^o)).
\]

By Theorem 3.3 in Stokey, Lucas, and Prescott (1989), the right hand side of (3) is a contraction: it defines a unique, continuous and differentiable value function \( v(g|y^o) \) for this interval of \( g \). (Differentiability follows from the differentiability of \( y(g|y^o) \)). Note that \( y(g|y^o) = \tilde{y}(g|y^o) \) for any \( g \in [g^2(y^o), y^o] \) and, by Lemma A.1, \( \tilde{y}(g|y^o) \in [g^2(y^o), y^o] \) for \( g \in [g^2(y^o), y^o] \). From the definition of \( \tilde{y}(g|y^o) \) and the discussion in Section 4 in the paper, it follows that \( u'(g) + \delta v'(g; y^o) = 1 \) for any \( g \in [g^2(y^o), y^o] \). In the rest of the state space we define the value function recursively. In \( [g^1(y^o), g^2(y^o)] \), if \( g^1(y^o) < g^2(y^o) \), the value function is defined as:

\[
v(g|y^o) = \frac{W + (1 - d)g - y(g^2(y^o)|y^o)}{n} + u(y(g^2(y^o)|y^o)) + \delta v(y(g^2(y^o)|y^o))
\]

where \( v(y(g^2(y^o)|y^o)) \) is well defined since \( y(g^2(y^o)|y^o) \in [g^2(y^o), y^o] \).

Lemma A.2. For \( g \in [g^1(y^o), y^o] \), \( u(g) + \delta v(g|y^o) \) is concave with slope larger or equal than 1.

Proof. If \( g^1(y^o) = g^2(y^o) \), the result is immediate. Assume therefore, \( g^1(y^o) < g^2(y^o) \).

In this case \( g^2(y^o) = y^*(\delta, d, n) \). For any \( g \in [g^1(y^o), g^2(y^o)] \), \( y(g; y^o) = y^*(\delta, d, n) \).

So we have \( u'(g|y^o) = (1 - d)/n \) implying: \( u'(g) + \delta v'(g|y^o) = u'(g) + \delta(1 - d)/n > 1 \) since \( g \leq g^2(y^o) = y^*(\delta, d, n) \). ■

Consider \( g < g^1(y^o) \). In \( [g_{-1}, g^1(y^o)] \) the value function is defined as:

\[
v(g|y^o) = u(W + (1 - d)g) + \delta v(W + (1 - d)g|y^o)
\]

where \( g_{-1} = \max \{0, [g^1(y^o) - W] / (1 - d) \} \). Assume that we have defined the value function in \( g \in [g_{-t}, g_{-(t-1)}] \) as \( v_{-t} \), for all \( t \) such that \( g_{-(t-1)} > 0 \). Then we can define \( v_{-(t+1)} \) as (5) in \( [g_{-(t+1)}, g_{-t}] \) with \( g_{-(t+1)} = [g_{-t} - W] / (1 - d) \).

Lemma A.3. For \( g \in [0, y^o] \), \( u(g) + \delta v(g|y^o) \) is concave with slope greater than or equal than 1.

Proof. We prove this by induction on \( t \). Consider now the interval \( [[g^1(y^o) - W] / (1 - d), g^1(y^o)] \).

In this range we have \( v'(g|y^o) = [u'(W + (1 - d)g) + \delta v'(W + (1 - d)g|y^o)](1 - d) \geq 1 - d \),
Finally consider Step 3. Let us consider Lemma A.4. For $g \leq g^3(y^o)$, $u(g) + \delta v(g|y^o)$ is concave with slope less than or equal to 1.

**Proof.** For $g \in \{y^o, g^3(y^o)\}$, $v'(g|y^o) = (1-d)/n$. Since $g \geq y^o \geq y^*(\delta, d, n)$, we have $u'(g) + \delta v'(g|y^o) = u'(g) + \delta (1-d)/n < 1$. Previous lemmas imply $u(g) + \delta v(g|y^o)$ is concave and has slope greater than or equal to 1 for $g \leq g^3(y^o)$. 

For $g \geq g^3(y^o)$, we must have $(1-d)g \in [y^o, g^3(y^o)]$. From the proof of Lemma A.5 we know that $u'(g) + \delta v'(g) < 1$ for $g \in [y^o, g^3(y^o)]$, so we have:

$$v'(g) = (1-d)[u'((1-d)g) + \delta v'((1-d)g)] < 1 - d$$

for $g > g^3(y^o)$. This fact implies that $u'(g) + \delta v'(g) < u'(g) + \delta (1-d)$ for any $g > g^3(y^o)$. Since $g^3(y^o) \geq \gamma(\delta, d)$ we have $u'(g) + \delta (1-d) < u'(\gamma(\delta, d)) + \delta (1-d) = 1$ for $g > g^3(y^o)$. It follows that $v^*(g)$ is has slope lower than 1 in $g > g^3(y^o)$. 

From Lemmata A1-A5 we conclude that $u(g) + \delta v(g|y^o)$ has a global maximum at any $g \in [g^3(y^o), y^o]$. 

**Step 3.** Define $x(g|y^o) = [W + (1-d)g - y(g|y^o)]/n$ and $i(g|y^o) = [y(g|y^o) - (1-d)g]/n$ as the levels of per capita private consumption and investment, respectively. Note that by construction, $x(g|y^o) \in [0, W/n]$. We now establish that $y(g|y^o)$, $x(g|y^o)$ and the associated value function $v(g|y^o)$ defined in the previous steps constitute an equilibrium. The fact that $v(g|y^o)$ describes the expected continuation value to an agent follows by construction. To see that $y(g|y^o)$ is an optimal reaction function given $v(g|y^o)$, note that an agent solves the following
We prove the result by contradiction. Suppose to the contrary there is a sequence of steady

Proposition 2. Proof of Proposition 2

where \( y(g) = y(g|\gamma^o) \). The investment function \( y(g|\gamma^o) \) satisfies the constraints of this problem

If \( g < g_1(\gamma^o) \), we have \( u'(y) + \delta v'(y) \geq 1 \) for all \( y \in [(1-d)g, W + (1-d)g] \), so \( y(g|\gamma^o) = W + (1-d)g \) is optimal. If \( g \geq g_1(\gamma^o) \), \( u'(y) + \delta v'(y) < 1 \) for all \( y \in [(1-d)g, W + (1-d)g] \), so

2 Proof of Proposition 2

Consider a sequence \( d^m \to 0 \). For each \( d^m \) there is at least an associated equilibrium \( y_m(g), v_m(g) \) with steady state \( y_m^o \). To prove the result we proceed in two steps. In Section 2.1 we prove that for any \( \xi > 0 \), there is a \( \tilde{m} \) such that for \( m > \tilde{m} \), \( J_1 d^m, n) \geq [u']^{-1} (1-\delta) - \xi \). In Section 2.2 we prove that the steady state of any equilibrium can not be larger than \( [u']^{-1} (1-\delta (1-d/n)) \). Since, as shown in Proposition 1, \( [u']^{-1} (1-\delta (1-d/n)) \) is an equilibrium steady state for any \( d \geq 0 \) and it converges to \( [u']^{-1} (1-\delta) \), we must have \( J_1 d, n) \to 0 \) as \( d \to 0 \). In Lemmata A.6 and A.7 presented in Section 2.2 we show that \( y'(0) \in (0,1) \) in a left neighborhood of the steady state \( y^o \) if \( y^o > [u']^{-1} (1-\delta (1-d)/n) \). Since all equilibrium steady states converge to \( [u']^{-1} (1-\delta) > [u']^{-1} (1-\delta/n) \), this implies that that convergence of \( g \) to the steady state is gradual in all equilibria if \( d \) is sufficiently small.

2.1 The lower bound

We prove the result by contradiction. Suppose to the contrary there is a sequence of steady states \( y_m^0 \), with associated equilibrium investment and value functions \( y_m(g), v_m(g) \), and an \( \xi > 0 \) such that \( y_m^0 < \gamma(0) - \xi \) for any arbitrarily large \( m \), where \( \gamma(d) = [u']^{-1} (1-\delta(1-d)) \). Define \( y_m^0(g) = y_m(g), \) and \( y_m^1(g) = y_m(y_m^0(g)) \) and consider a marginal deviation from the steady state
from $y_m^0$ to $y_m^0 + \Delta$. By the irreversibility constraint we have $y_m(g) \geq (1-d^m)g$. Using this property and the fact that $y_m^0$ is a steady state, so $y_m^0(y_m^0) = y_m^0$, we have:

$$y_m(y_m^0 + \Delta) - y_m(y_m^0) \geq (1-d^m)(y_m^0 + \Delta) - y_m^0 = (1-d^m)\Delta - d^m y_m^0$$

This implies that, as $m \to \infty$, for any given $\Delta$: $[y_m(y_m^0 + \Delta) - y_m^0] / \Delta \geq 1 + o_1(d^m)$ where $o_1(d^m) \to 0$ as $m \to 0$. We now show with an inductive argument that a similar property holds for all iterations $y_m^j(y_m^0)$. Assume we have shown that: $[y_m^{j-1}(y_m^0 + \Delta) - y_m^0] / \Delta \geq 1 + o_{j-1}(d^m)$ where $o_{j-1}(d^m) \to 0$ as $m \to 0$. We must have: $y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0(y_m^0) \geq (1-d^m)y_m^{j-1}(y_m^0 + \Delta) - y_m^0$. We therefore have: $y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0 \geq y_m^{j-1}(y_m^0 + \Delta) - y_m^0 - d^m y_m^{j-1}(y_m^0 + \Delta)$ so we have:

$$\frac{y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0}{\Delta} \geq \frac{y_m^{j-1}(y_m^0 + \Delta) - y_m^0}{\Delta} - \frac{d^m y_m^{j-1}(y_m^0 + \Delta)}{\Delta} \geq 1 + o_j(d^m) \quad (7)$$

where $o_j(d^m) = o_{j-1}(d^m) - \frac{d^m y_m^{j-1}(y_m^0 + \Delta)}{\Delta}$, so $o_j(d^m) \to 0$ as $m \to 0$.

We can write the value function after the deviation to $y_m^0 + \Delta$ as:

$$V(y_m^0 + \Delta) = \sum_{j=0}^{\infty} \delta^{j-1} \left[ W + \frac{(1-d^m)y_m^{j-1}(y_m^0 + \Delta) - y_m^0}{n} + u(y_m^0(y_m^0 + \Delta)) \right]$$

For any given function $f(x)$, define $\Delta f(x) = f(x + \Delta) - f(x)$. We can write:

$$\Delta V(y_m^0) / \Delta = \sum_{j=0}^{\infty} \delta^{j-1} \left[ \frac{(1-d^m)\Delta y_m^{j-1}(y_m^0) / \Delta - \Delta y_m^j(y_m^0) / \Delta}{n} + [u'(y_m^0) + o(\Delta)] y_m^j(y_m^0) / \Delta \right]$$

$$\geq \sum_{j=0}^{\infty} \delta^{j-1} \left[ \frac{(1-d^m)(1+o_{j-1}(d^m)) - (1+o_j(d^m))}{n} + [u'(y_m^0) + o(\Delta)] (1 + o_j(d^m)) \right] \quad (8)$$

where $o(\Delta) \to 0$ as $\Delta \to 0$. In the first equality we use the fact that if we choose $\Delta$ small, since $y_m(g)$ is continuous, $\Delta y_m^j(y_m^0)$ is small as well. This implies that

$$\frac{(u(y_m^0(y_m^0 + \Delta)) - u(y_m^0(y_m^0))) / \Delta}{y_m^0(y_m^0 + \Delta) - y_m^0(y_m^0)}$$

converges to $u'(y_m^0(y_m^0))$ as $\Delta \to 0$. The inequality in 8 follows from (7). Given $\Delta$, as $m \to \infty$, we therefore have $\lim_{m \to \infty} \Delta V(y_m^0) / \Delta \geq \frac{u'(y_m^0) + o(\Delta)}{1 - \delta}$. We conclude that for any $\varepsilon > 0$, there must be a $\Delta_0$ such that for any $\Delta \in (0, \Delta_0)$ there is a $m_\Delta$ guaranteeing that $\Delta V(y_m^0) / \Delta \geq \frac{u'(y_m^0)}{1 - \delta} - \varepsilon$ for $m > m_\Delta$. After a marginal deviation to $y_m^0 + \Delta$, therefore, the change in agent’s objective function is:

$$u'(y_m^0) + \delta \Delta V(y_m^0) / \Delta - 1 \geq \frac{u'(y_m^0)}{1 - \delta} - \delta \varepsilon - 1 \quad (9)$$

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for $m$ sufficiently large. A necessary condition for the un-profitability of a deviation from $y^0_m$ to $y^0_m + \Delta$ is therefore: $y^0_m \geq [u']^{-1}(1 - \delta + \varepsilon(1 - \delta))$. Since $\varepsilon$ can be taken to be arbitrarily small, for an arbitrarily large $m$, this condition implies $y^0_m \geq \bar{y}(0) - \xi/2$, which contradicts $y^0_m < \bar{y}(0) - \xi$. We conclude that $y^0_{IR}(\delta,d,n) \rightarrow \bar{y}(0)$ as $d \rightarrow 0$.

2.2 The upper bound

Suppose to the contrary that there is a stable steady state at $y^\circ > [u']^{-1}(1 - \delta (1 - d/n))$. We must have $y^\circ \in [u']^{-1}(1 - \delta (1 - d/n))W/d$, since it is not feasible for a steady state to be larger than $W/d$. Consider a left neighborhood of $y^\circ$, $N_\varepsilon(y^\circ) = (y^\circ - \varepsilon, y^\circ)$. The value function can be written in $g \in N_\varepsilon(y^\circ)$ as:

$$v(g) = u(y(g)) + \delta v(y(g)) - y(g) + \frac{W + (1 - d)g}{n} + (1 - 1/n)y(g)$$

(9)

where $y(g)$ is the equilibrium strategy associated to $y^\circ$. In $N_\varepsilon(y^\circ)$ the constraint $y \geq \frac{1-d}{n}g + \frac{\delta}{n}y(g)$ cannot be binding (else we would have $y(g) = (1 - d)g$, but this is not possible in a neighborhood of $y^\circ > 0$). We consider two cases.

Case 1. Suppose first that $y^\circ < W/d$. We must therefore have that $y(g) < W + (1 - d)g$ in $N_\varepsilon(y^\circ)$, so the constraint $y \leq \frac{W + (1 - d)g}{n} + \frac{\delta}{n}y(g)$ is not binding. The solution is in the interior of the constraint set of (6), and the objective function $u(y(g)) + \delta v(y(g)) - y(g)$ is constant for $g \in N_\varepsilon(y^\circ)$.

Lemma A.6. If $y^\circ > [u']^{-1}(1 - \delta (1 - d)/n)$, then there is a left neighborhood $N_\varepsilon(y^\circ)$ in which $y(g)$ is not constant.

Proof. Suppose to the contrary that, for any $N_\varepsilon(y^\circ)$, there is an interval in $N_\varepsilon(y^\circ)$ in which $y(g)$ is constant. Using the expression for $v(g)$ presented above, we must have $v'(g) = (1 - d)/n$ for any $g$ in this interval. Since $N_\varepsilon(y^\circ)$ is arbitrary, then we must have a sequence $g^m \rightarrow y^\circ$ such that $v'(g^m) = (1 - d)/n \forall m$. We can therefore write:

$$\lim_{\Delta \rightarrow 0} \frac{v(y^\circ) - v(y^\circ - \Delta)}{\Delta} = \lim_{\Delta \rightarrow 0} \lim_{m \rightarrow \infty} \frac{v(g^m) - v(g^m - \Delta)}{\Delta} = \lim_{m \rightarrow \infty} \lim_{\Delta \rightarrow 0} \frac{v(g^m) - v(g^m - \Delta)}{\Delta} = \frac{1 - d}{n}$$

where the second equality follows from the continuity of $v(g)$. This implies that $v^\circ - (y^\circ)$, left derivative of $v(g)$ at $y^\circ$, is well defined and equal to $\frac{1 - d}{n}$. Consider now a marginal reduction of $g$ at $y^\circ$. The change in utility is (as $\Delta \rightarrow 0$):

$$\Delta U_n(y^\circ) = u(y^\circ - \Delta) - u(y^\circ) + \delta [v(y^\circ - \Delta) - v(y^\circ)] + \Delta$$

$$= \left[1 - \left(u'(y^\circ) + \delta \frac{1 - d}{n}\right)\right] \Delta$$
In order to have $\Delta U(y^o) \leq 0$, we must have $u'(y^o) + \delta(1-d)/n \geq 1$. This implies $y^o \leq [u']^{-1} (1 - \delta (1-d)/n)$, a contradiction. Therefore, if there is stable steady state at $y^o > [u']^{-1} (1 - \delta (1-d)/n)$, then $y(g)$ is not constant in $N_\varepsilon(y^o)$. ■

Lemma A.6 implies that there is a left neighborhood $N_\varepsilon(y^o)$ in which $u(g) + \delta v(g) - g$ is constant if $y^o > [u']^{-1} (1 - \delta (1-d)/n)$ (since otherwise $y(g)$ would be constant). Moreover, since $y^o$ is a stable steady state and $y(g)$ is strictly increasing, $g \in N_\varepsilon(y^o)$ implies $y(g) \in N_\varepsilon(y^o)$ for any open left neighborhood $N_\varepsilon(y^o) = (y^o - \varepsilon, y^o) \subset N_\varepsilon(y^o)$. These observations imply:

Lemma A.7. If $y^o > [u']^{-1} (1 - \delta (1-d)/n)$, then there is a left neighborhood $N_\varepsilon(y^o)$ in which

$$y'(g) = \frac{n}{n-1} \left( \frac{1 - u'(g)}{\delta} - \frac{1-d}{n} \right)$$

(10)

Proof. There is a $N_\varepsilon(y^o)$ and a constant $K$ such that $\delta v(g) = K + g - u(g)$ for $g \in N_\varepsilon(y^o)$. Hence $v(g)$ is differentiable in $N_\varepsilon(y^o)$. Moreover, $y(g) \in N_\varepsilon(y^o)$ for all $g \in N_\varepsilon(y^o)$. Hence $u(y(g)) + \delta v(y(g)) - y(g)$ is constant in $g \in N_\varepsilon(y^o)$ as well. These observations and the definition of $v(g)$ imply that $v'(g) = \frac{1-d}{n} + (1 - \frac{1}{n}) y'(g)$ in $N_\varepsilon(y^o)$. Given that $u'(g) + \delta v'(g) = 1$ in $g \in N_\varepsilon(y^o)$, we must have: $u'(g) + \delta v'(g) = u'(g) + \delta [\frac{1-d}{n} + (1 - \frac{1}{n}) y'(g)] = 1$ which implies (10) for any $g \in N_\varepsilon(y^o)$. ■

Let $g^m$ be a sequence in $N_\varepsilon(y^o)$ such that $g^m \to y^o$. We must have

$$y^-(y^o) = \lim_{\Delta \to 0} \frac{y(y^o) - y(y^o - \Delta)}{\Delta} = \lim_{\Delta \to 0} \lim_{m \to \infty} \frac{y(g^m) - y(g^m - \Delta)}{\Delta} = \lim_{m \to \infty} \lim_{\Delta \to 0} \frac{y(g^m) - y(g^m - \Delta)}{\Delta} = \frac{n}{n-1} \left( \frac{1 - u'(y^o)}{\delta} - \frac{1-d}{n} \right)$$

(11)

where $y^-(y^o)$ is the left derivative of $y(g)$ at $y^o$, the second equality follows from continuity and the last equality follows from Lemma A.7 and the fact that under the starting assumption we have $y^o > [u']^{-1} (1 - \delta (1-d)/n) > [u']^{-1} (1 - \delta (1-d)/n)$. Consider a state $(y^o - \Delta)$. For $y^o$ to be stable we need that for any small $\Delta$:

$$y(y^o - \Delta) \geq y^o - \Delta = y(y^o) + (y^o - \Delta) - y^o$$

where the equality follows from the fact that $y(y^o) = y^o$. As $\Delta \to 0$, this implies $y^-(y^o) \leq 1$ in $N_\varepsilon(y^o)$. By (11), we must therefore have: $\frac{n}{n-1} \left( \frac{1-u'(y^o)}{\delta} - \frac{1-d}{n} \right) \leq 1$. This implies: $y^o \leq [u']^{-1} (1 - \delta (1-d)/n)$, a contradiction.

Case 2. Assume now that $y^o = W/d$ and consider first the case in which it is a strict local maximum of the objective function $u(y) + \delta v(y) - y$. In this case in a left neighborhood $N_\varepsilon(y^o)$, we have that the upper bound $y \leq W + (1-d)g + \frac{n-1}{n} y(g)$ is binding: implying $y(g) = W + (1-d)g$.
Proposition 4. We must therefore have a sequence of points \( g^m \to y^o \) such that \( g^m = y(g^{m-1}) \) and \( y(g^m) = W + (1-d)g^m \forall m \). Given this, we can write:

\[
v(g^m) = u(g^{m+1}) + \delta v(g^{m+1}) = u(g^{m+1}) + \delta \left[ u(g^{m+2}) + \delta v(g^{m+2}) \right]
\]

\[
= \sum_{j=0}^{\infty} \delta^j u(W + (1-d)g^{m+j})
\]

We therefore must have that \( v(g^m) \) is differentiable and \( \delta v'(g^m) = \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j}) \). Since \( u'(g^m) + \delta v'(g^m) \geq 1 \), we have \( u'(g^m) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j}) \geq 1 \) for all \( m \). Consider the limit as \( m \to \infty \). Since \( u'(g) \) is continuous and \( g^m \to y^o \), we have:

\[
1 \leq \lim_{m \to \infty} \left[ u'(g^m) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j}) \right]
\]

\[
= u'(y^o) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(y^o) = \frac{u'(y^o)}{1 - \delta(1-d)}
\]

This implies \( y^o \leq [u']^{-1}(1 - \delta(1-d)) < [u']^{-1}(1 - \delta (1 - d/n)) \), a contradiction. Assume now that \( y^o = W/d \), but it is not a strict maximum of \( u(y) + \delta v(y) - y \) in any left neighborhood. It must be that \( u(y) + \delta v(y) - y \) is constant in some left neighborhood \( N_{\epsilon}(y^o) \). If this were not the case, then in any left neighborhood we would have an interval in which \( y(g) \) is constant, but this is impossible by Lemma A.6. But then if \( u(y) + \delta v(y) - y \) is constant in some \( N_{\epsilon}(y^o) \), the same argument as in Step 1 implies a contradiction. ■

3 Proof of Proposition 4

**Proposition 4.** For any \( d > 0 \) and \( n \), there is a \( \overline{\delta} < 1 \) such that the most efficient SPE path in a RIE and the most efficient SPE path in a IIE coincide with the Pareto efficient investment path for any \( \delta > \overline{\delta} \). Hence, neither the most efficient SPE path in a RIE nor the most efficient SPE path in a IIE is characterized by gradualism for any \( \delta > \overline{\delta} \).

We first show that there is a \( \delta_1 < 1 \), such that for \( \delta > \delta_1 \) the efficient path is a SPE path in an irreversible investment economy. To this goal, we first define the equilibrium strategies and establish some key properties. Let \( y^M(g; d, \delta) \), \( v^M(g; d, \delta) \) be, respectively, the investment function and the value function of the Markov equilibrium with the lowest steady state characterized in Proposition 2 when the discount factor is \( \delta \) and the rate of depreciation is \( d \). Let \( g^M(d, \delta) = [u']^{-1}(1 - \delta(1-d)/n) \) be the associated steady state. It is easy to see that, for any \( d \) and \( n \), \( g^M(d, \delta) < y^*_p(\delta, d, n) \) for all \( \delta \in [0, 1] \). Define \( y^M_j(g; d, \delta) \) recursively with \( y^0_j(g; d, \delta) = g \) and \( y^M_j(g; d, \delta) = y^M(y^M_{j-1}(g; d, \delta); d, \delta) \). For any \( g \), \( y^M_j(g; d, \delta) \to g^M(d, \delta) \) as \( j \to \infty \). It follows that \( \lim_{\delta \to 1} \left[ (1 - \delta) v^M(g; d, \delta) \right] = (W - dg^M(d, 1))/n + u(g^M(d, 1)) \). Let \( y^F(g; d, \delta) \) be the
efficient investment function characterized in Section 3 with steady state $g^P(d,\delta) = y^*_P(\delta,d,n)$, and let $v^P(g;d,\delta)$ be the associated expected utility for a player. Similarly, it is easy to see that $\lim_{\delta \to 1} \left[ (1 - \delta) v^P(g;d,\delta) \right] = \left( W - dg^P(d,1) \right) / n + u(g^P(d,1))$, where $y^P(g;d,\delta)$ be the efficient investment function characterized in Section 3 with steady state $g^P(d,\delta) = y^*_P(\delta,d,n)$). It follows that $\lim_{\delta \to 1} \left[ (1 - \delta) v^P(g;d,\delta) \right] > \lim_{\delta \to 1} \left[ (1 - \delta) v^M(g;d,\delta) \right]$.

Associated to an aggregate investment function $y^I(g;d,\delta), l = \{M, P\}$, we have the individual contribution function: $i^I(g;d,\delta) = \left[ y^I(g;d,\delta) - (1 - d)g \right] / n$. To construct the equilibrium, consider the following trigger strategies. If $g^* = y^P(g_0;d,\delta)$ for all $\tau \leq t$, then $i^I(g;\tau;d,\delta) = i^P(g;d,\delta)$, where $i^I(g_t)$ is the investment at time $t$ of an agent. If $\exists \tau \leq t$ such that $g_{\tau} \neq y^P(g_0;d,\delta)$, then $i^I(g_t) = i^M(g;d,\delta)$. Note that, by construction, deviations are not profitable after a $\tau$ such that $g_{\tau} \neq y^P(g_0;d,\delta)$. For the remaining histories note that the average utility of a deviating agent must converge to $(1 - \delta) v^M(g;d,\delta) < (1 - \delta) v^P(g;d,\delta)$, so there must be a $\delta_1 < 1$, such that for $\delta > \delta_1$ no deviation is profitable.

The result that we also have a $\delta_2 < 1$, such that for $\delta > \delta_2$ the efficient path is a SPE path in a reversible investment economy can be proven analogously. From Battaglini et al. [2012], we know that there is a Markov equilibrium $\tilde{y}^M(g;d,\delta), \tilde{v}^M(g;d,\delta)$ with steady state $\tilde{y}^M(d,\delta) \leq [u']^{-1} (1 - \delta(1 - d)/n)$, and so strictly lower than the steady state $g^P(d,1)$ of the planner’s solution for all $\delta \in [0,1]$. Proceeding exactly as above we can see that $\lim_{\delta \to 1} \left[ (1 - \delta) v^P(g;d,\delta) \right] > \lim_{\delta \to 1} \left[ (1 - \delta) \tilde{v}^M(g;d,\delta) \right]$. Associated to an aggregate investment function $\tilde{y}^M(g;d,\delta)$ we define as above the individual contribution function: $\tilde{i}^M(g;d,\delta) = \left[ \tilde{y}^M(g;d,\delta) - (1 - d)g \right] / n$. To construct the equilibrium, consider the following trigger strategies. If $g^* = y^P(g_0;d,\delta)$ for all $\tau \leq t$, then $\tilde{i}^M(g;\tau;d,\delta) = i^P(g;d,\delta)$, where $i^I(g_t)$ is the investment at time $t$ of an agent. If $\exists \tau \leq t$ such that $g_{\tau} \neq y^P(g_0;d,\delta)$, then $\tilde{i}^M(g_t) = i^M(g;d,\delta)$. Note that, by construction, deviations are not profitable after a $\tau$ such that $g_{\tau} \neq y^P(g_0;d,\delta)$. For the remaining histories note that the average utility of a deviating agent must converge to $(1 - \delta) v^M(g;d,\delta) < (1 - \delta) v^P(g;d,\delta)$, so there must be a $\delta_2 < 1$, such that for $\delta > \delta_2$ no deviation is profitable. Given this, the statement of the proposition follows immediately by defining $\delta = \max(\delta_1, \delta_2)$. ■